Knowledge Representation and Reasoning with First Order Logic

Deepak Khemani

The Syllabus

Introduction: Overview and Historical Perspective

First Order Logic: A logic with quantified variables.

Module 1 (2 hours): Syntax, Semantics, Entailment and Models, Proof Systems, Knowledge Representation.

Module 2 (2 hours): Skolemization, Unification, Deductive Retrieval, Forward Chaining, Backward Chaining

Module 3 (2 hours): Resolution Refutation in FOL, Horn Clauses and Logic Programming

Module 4 (2 hours): Variations on FOL

Text book

Deepak Khemani. A First Course in Artificial Intelligence (Chapters 12 & 13), McGraw Hill Education (India), 2013.

Some definitions of Artificial Intelligence

We call programs intelligent if they exhibit behaviors that would be regarded intelligent if they were exhibited by human beings.

– Herbert Simon

Physicists ask what kind of place this universe is and seek to characterize its behavior systematically. Biologists ask what it means for a physical system to be living. We in AI wonder what kind of information-processing system can ask such questions. – Avron Barr and Edward Feigenbaum

AI is the study of techniques for solving exponentially hard problems in polynomial time by exploiting knowledge about the problem domain.

– Elaine Rich

AI is the study of mental faculties through the use of computational models.

– Eugene Charniak and Drew McDermott

Machines with Minds of their Own

"The fundamental goal of Artificial Intelligence research is not merely to mimic intelligence or produce some clever fake.

Not at all.

"AI" wants the genuine article: machines with minds, in the full and literal sense.

This is not science fiction, but real science, based on a theoretical conception as deep as it is daring: namely, we are at root, computers ourselves.

That idea – the idea that thinking and computing are radically the same – is the idea of this book."

John Haugeland in "AI: The Very Idea"

Physical Symbol Systems

Symbol : A perceptible something that stands for something else.
- alphabet symbols, numerals, road signs, musical notation

Symbol System: A collection of symbols – a pattern
- words, arrays, lists, even a tune

Physical Symbol System: That obeys laws of some kind, a formal system
- long division, an abacus, an algorithm

The Physical Symbol System Hypothesis

"A physical symbol system has the **necessary** and **sufficient** means for general intelligent action."

— Allen Newell and Herbert A. Simon

The ability to manipulate symbols - Symbolic AI / Classical AI

Good Old Fashioned Artificial Intelligence (GOFAI)

– John Haugeland in *AI: The Very Idea*

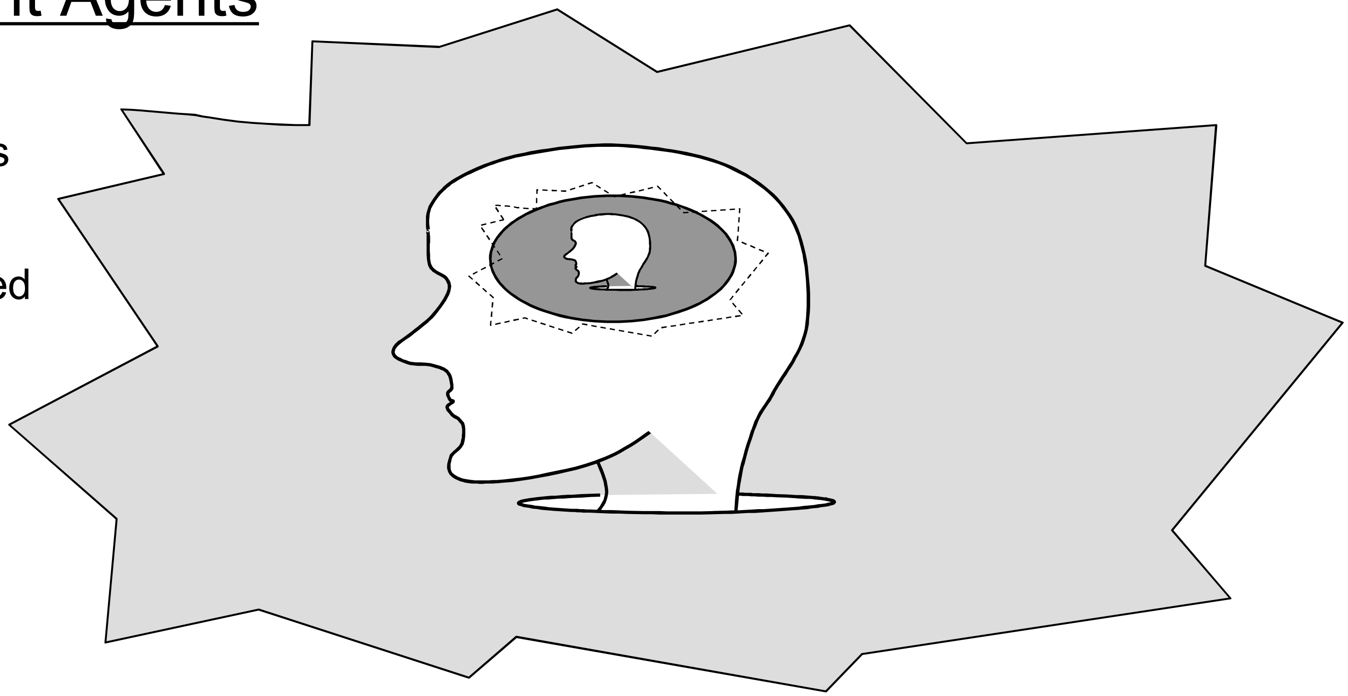
Intelligent Agents

Persistent

Autonomous

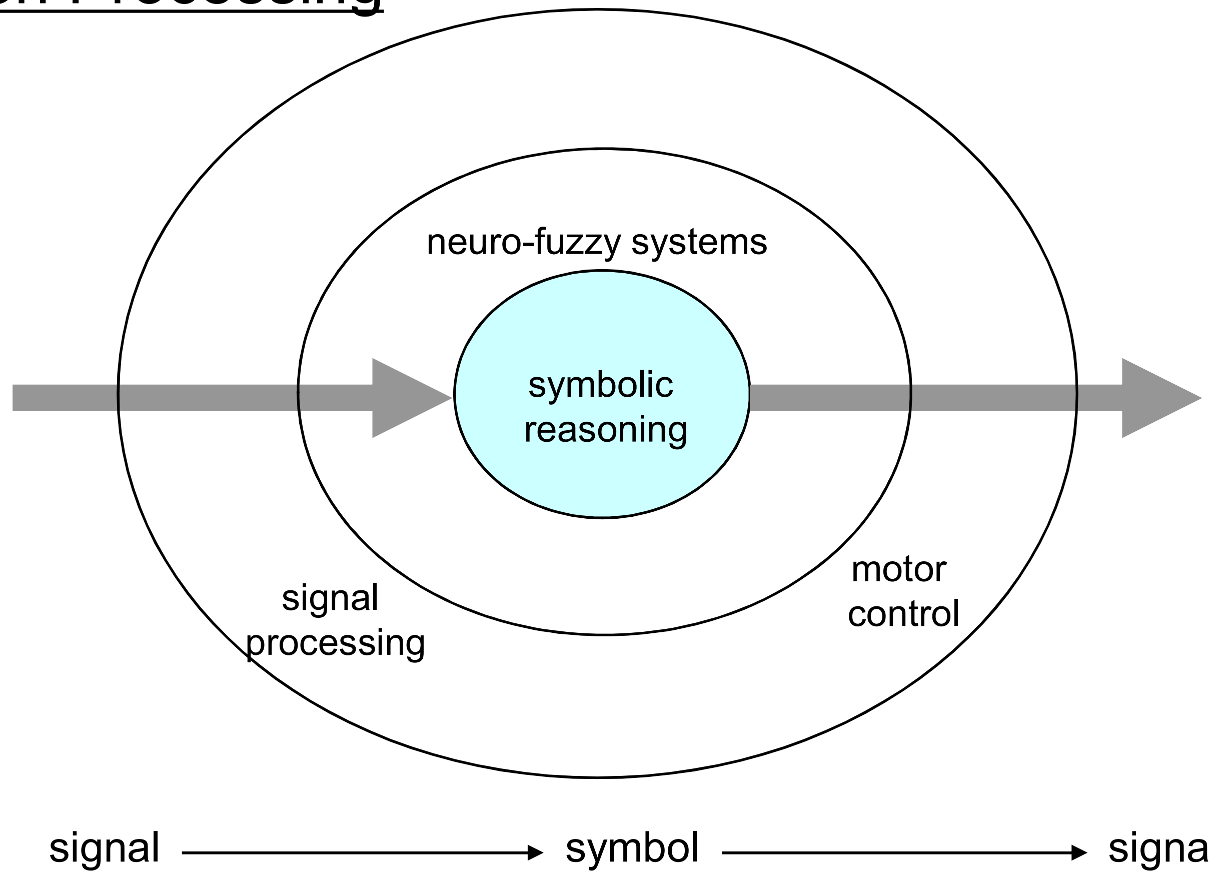
Proactive

Goal Directed

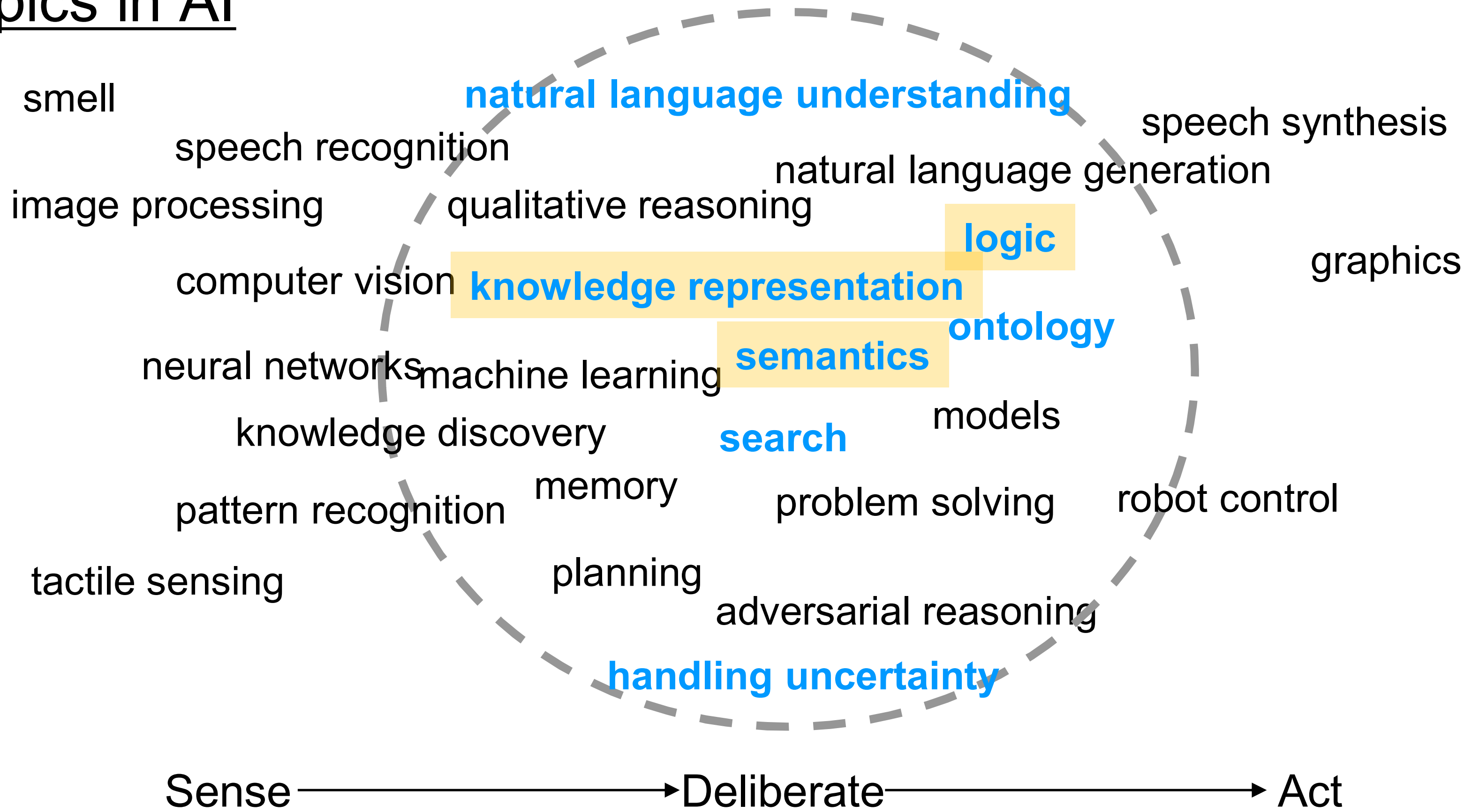


An intelligent agent in a world carries a model of the world in its "head". The model maybe an abstraction. A self aware agent would model itself in the world model. Deeper awareness may require that the agent represent (be aware of) itself modeling the world. (From A First Course in AI – Deepak Khemani)

Information Processing



Topics in AI



Source: Deepak Khemani, A First Course in Artificial Intelligence

Knowledge (and Memory)

Knowledge — To know. Humans deal with knowledge of many kinds. We have models of the world we live in. We have models of ourselves in the world. We have knowledge of society, knowledge of facts, knowledge of how to do things etc…

Ontology — Knowledge of being. How the world is. Of concepts and the relations between them.

Epistemology — Knowledge about what is true in the world.

Knowledge Based Systems — Usually refers to systems that employ domain specific problem solving knowledge in some form.

Memory — Our repository of knowledge. In human beings memory is dynamic. We continuously learn.

Knowledge and Reasoning – necessary for intelligence

What does the agent know

and

what else does the agent know as a consequence of what it knows?

Representation

Semiotics: A symbol is something that stands for something else

Examples.

- The “number” seven can be represented in many different ways.
- Road signs – curves, pedestrians, schools, U-turns, eating places…

All languages are semiotic systems

Biosemiotics: How complex behaviour emerges when simple systems interact with each other through signs.

Reasoning

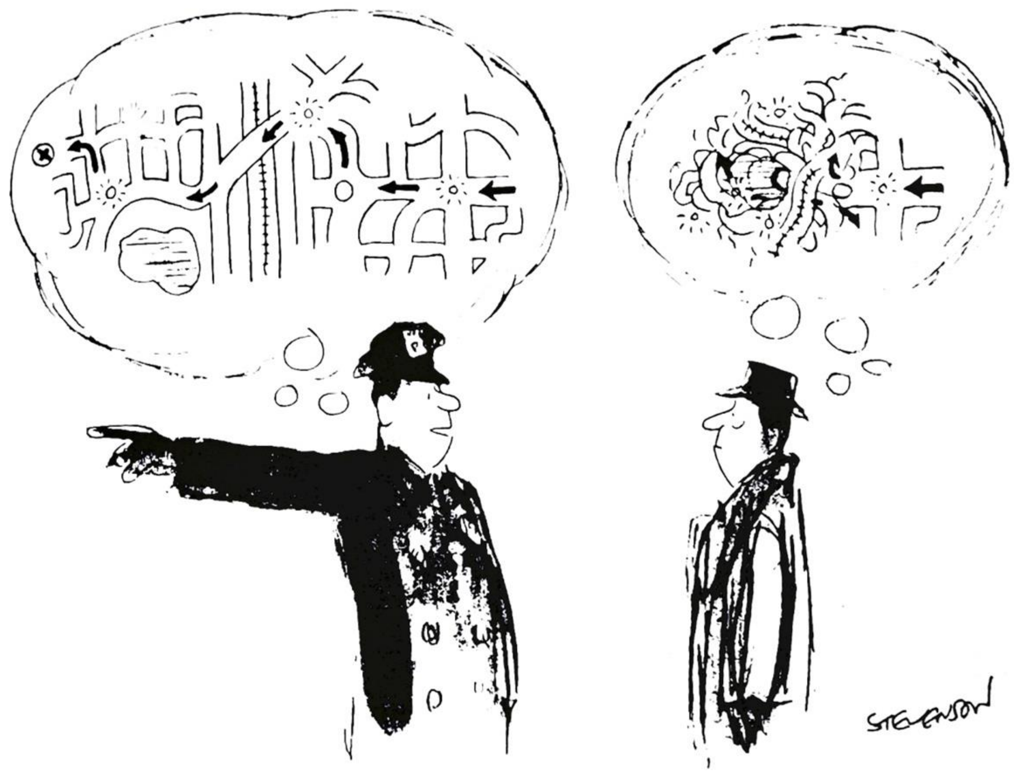
The manipulation of symbols in a meaningful manner.

Maths is replete with *algorithms* we use –

- Addition and multiplication of multi-digit numbers
- Long division
- Solving systems of linear equations
- Fourier transforms, convolution…

Natural Language

Richness
Ambiguity
Verbosity
Impreciseness



Drawing by Stevenson; © 1976 The New Yorker Magazine, Inc.

John Sowa: Conceptual Structures

Fig. 1.5 Using language to express a mental model

The Syllogism

The Greek syllogism embodies the notion of **formal logic**.

An argument is valid if it conforms to a valid **form**

All men are mortal	All cities are congested	All politicians are honest
Socrates is a man	Chennai is a city	Sambit is a politician
Socrates is mortal	Chennai is congested	Sambit is honest

The Socratic argument

In a **valid** argument
IF the premises are true
THEN the conclusions are **necessarily** true

Formal Logic

Logic is a formal system

Logical reasoning is only concerned ONLY with the FORM of the argument, and not with CONTENT.

If the form is valid AND If the antecedents are true
THEN the conclusion is true.

Thus, the conclusion holds only if the antecedents are true.

Logic does not concern itself with the truth of antecedents OR what the sentences are talking about (content).

Indian Philosophy

Logic in India goes back to the art of debating. The six well known schools of philosophy are:

- *Sãṃkhya* (Kapila around 500 B.C.),
- *Miṃãsṃã* (Jaimini around 300 B.C.),
- *Nyãya* (Akṣapāda Gautama, 2nd century B.C.),
- *Yoga* (Patanjali, possibly 2nd century B.C. or later),
- *Vaiseṣika* (Kanada, 6th century B.C.), and
- *Vedãnta* (also called *uttara Miṃãsṃã*, the word *Vedãnta* means the end of all knowledge, were composed starting the 9th century B.C. in the *Upanishads,* is credited to Veda Vyāsa, also the author of the Indian epic, *Mahābhārata*).

Nyãyasũtra by Gautama

The *Nyãyasũtra* by Gautama (or Gotama) written in second century B.C. was concerned with the knowledge of sixteen categories (Sinha and Vidyabhusana, 1930),

1. means of valid knowledge (*pramana*)
2. objects of valid knowledge (*prameya*)
3. doubt (*samshaya*)
4. purpose (*prayojana*)
5. example (*drstanta)*
6. conclusion (*siddhanta*)
7. the constituents of a syllogism (*avayava*)
8. argumentation (*tarka*)
9. ascertainment (*nirnaya*)
10. debate (*vada*)
11. disputations (*jalpa*)
12. destructive criticism (*vitanda*)
13. fallacy (*hetvabhasa*)
14. quibble (*chala*)
15. refutations (*jāti*), and
16. points of the opponent's defeat (*nigrahasthana*).

http://en.wikipedia.org/wiki/Aksapada_Gautama

Reasoning

Of the four possible sources of knowledge –

•perception (*pratyaksha*)

•inference (*anumãna*)

•comparison (*upamãna)*, and

•verbal testimony (*shabda,* which could be of God from the *vedãs*, or of a trustworthy human!)

– the mode of *inference* is concerned with logical necessity.

This is reflected in the form of the argument now known as the five step syllogism.

The 5 Step Syllogism

The *Nyãyasũtra* describes the structure (Mohanty, 2000) of a good argument as a five step process.

1. a statement of the thesis (*Pratijñā*): *there is fire on the mountain*
2. a statement of reason (*Hetu*): *because there is smoke on the mountain*
3. an example of the underlying rule (*Udahãrana*): *where there is smoke there is fire, like the culinary hearth*
4. a statement that (*Upãnaya*): *this case is like that*
5. finally the assertion of the thesis proven (*Nigamana*): *therefore the mountain is on fire*

The derived piece of knowledge is known as *anumãna* (after cognition).

Logical *Arguments*

This is in contrast to the three step syllogism exemplified by the Socratic argument. It has also been observed by Müller (1853; 1859) that the Indian philosophers used the five step reasoning only when the task was to convince others about their conclusions. When the task was to infer something for oneself the simpler three step process was used, as follows.

1. There is smoke on the mountain
2. Wherever there is smoke there is fire
3. Therefore, there is fire on the mountain

This is precisely the form of reasoning, the Aristotelian syllogism, which is fundamental to western logic.

Formal Logics

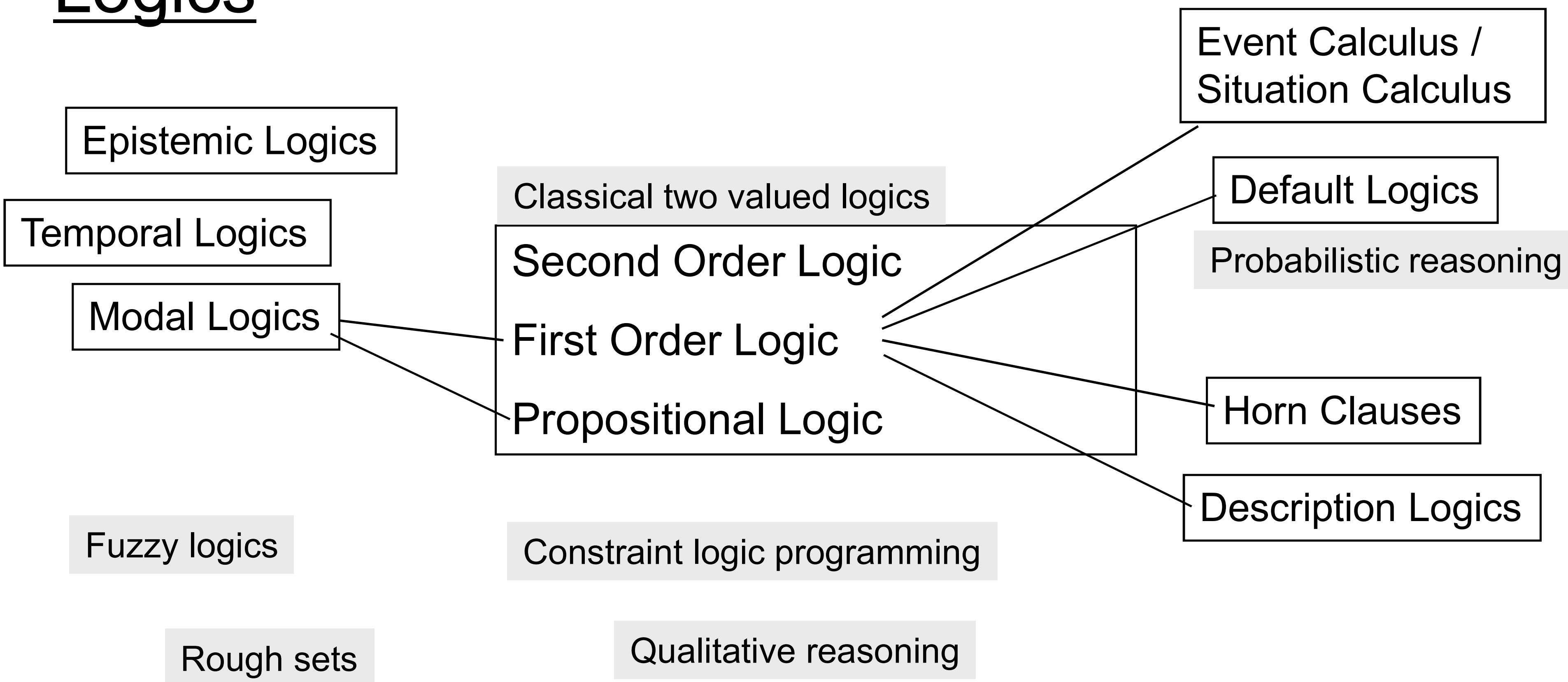
Logics are formal languages with well defined rules for manipulation of representations.

A *knowledge base* (KB) is a set of sentences in a given logic language.

The family of logics vary on expressivity.

More expressivity comes at the cost of increasing computational complexity.

Logics



Event Calculus / Situation Calculus
Epistemic Logics
Classical two valued logics
Default Logics
Temporal Logics
Second Order Logic
Probabilistic reasoning
Modal Logics
First Order Logic
Propositional Logic
Horn Clauses
Description Logics
Fuzzy logics
Constraint logic programming
Rough sets
Qualitative reasoning

Propositional Logic

Which of the arguments are valid arguments?

- If the earth were spherical, it would cast curved shadows on the moon. It casts curved shadows on the moon. So it must be spherical.

- If he used good bait and the fish weren't smarter than he was, then he didn't go hungry. But he used good bait and he did go hungry, **SO** the fish must have been smarter than he was.

First Order Logic

One of Tinker, Tailor, Soldier, or Spy is the culprit. The culprit stole the document. Tinker and Soldier did not steal the document. If Tailor or Spy is the culprit, then the document must be in Paris.

Given the above facts show that the following sentence is true

"The document is in Paris."

Notion of variables and quantifiers over variables

Timeless, changeless, a logic of relations between elements of sets

Description Logics

A progressive high tech company is one with at least five women on its board of directors and one in which all the employees have technical degrees where the minimum salary is 100000.

A progressive high tech company is a tech company.

A family of logics of noun phrases

The formal basis of ontologies

Default Reasoning

If Tweety is a bird then conclude that Tweety can fly,

because even though there exist birds, for example the ostrich, that cannot fly, in general most birds fly.

In the real world an intelligent agent has to make inferences even with incomplete information. In such a scenario one has to make use of what is generally true in a typical scenario.

New information may contradict and defeat the conclusion.

The Event Calculus

Jogesh made a cup of tea and left it on the table. Meanwhile Smita saw the cup of tea and drank it. When Jogesh came back he saw that the cup was empty.

Reasoning about time, action, and change.

Epistemic reasoning

Jogesh made a cup of tea and left it on the table. Meanwhile Smita saw the cup of tea and drank it. When Jogesh came back he saw that the cup was empty.

He concluded that Smita had polished off his cup of tea.
Smita knew that Jogesh knew that she drank the tea.

Knowledge and belief of agents

Introduction to First Order Logic

Logic: Syntax

A formal axiomatic system

Alphabet → (Formal) Language *L* = A set of sentences

Axioms (or Premises): a subset *S* of *L* = the knowledge base (KB)

Rules of Inference: a set of rules that allow more sentences from L to be added to the KB

Goal: Given a KB *S* can a new sentence α be added to the KB by repeated application of some rules of inference?

If yes, then we say that α is provable.

$$KB \vdash \alpha$$

Some common rules of inference

From	$\alpha \supset \beta$
and	$\underline{\alpha}$
Infer	β
Modus Ponens (MP)	

From	$\alpha \supset \beta$
and	$\underline{\sim\beta}$
Infer	$\sim\alpha$
Modus Tollens (MT)	

From	α
and	$\underline{\beta}$
Infer	$\alpha \wedge \beta$
Conjunction (C)	

From	$\alpha \vee \beta$
and	$\underline{\sim\alpha}$
Infer	β
Disjuncuntive Syllogism (DS)	

From	$\underline{\alpha}$
Infer	$\alpha \vee \beta$
Addition (A)	

From	$\underline{\alpha \wedge \beta}$
Infer	α
Simplification (S)	

From	$\alpha \supset \beta$
and	$\underline{\beta \supset \gamma}$
Infer	$\alpha \supset \gamma$
Hypothetical Syllogism (HS)	

From	$(\alpha \supset \beta) \wedge (\gamma \supset \delta)$
and	$\underline{\alpha \vee \gamma}$
Infer	$\beta \vee \delta$
Constructive Dilemma (CD)	

From	$(\alpha \supset \beta) \wedge (\gamma \supset \delta)$
and	$\underline{\sim\beta \vee \sim\delta}$
Infer	$\sim\alpha \vee \sim\gamma$
Destructive Dilemma (DD)	

Rules of Substitution

A rule of substitution allows one to replace one sentence with another. This is possible when one sentence is logically equivalent to another. As an example let us look at the following equivalence.

$$((\alpha \supset \beta) \equiv (\neg\alpha \vee \beta))$$

If the above equivalence is a tautology, then the sentence $(\alpha \supset \beta)$ will always take the same truth value as the sentence $(\neg\alpha \vee \beta)$. Hence either of the two could be replaced by the other without any loss. We can verify that the equivalence is a tautology by constructing a truth table.

α	β	$(\alpha \supset \beta)$	$\neg\alpha$	$(\neg\alpha \vee \beta)$	$((\alpha \supset \beta) \equiv (\neg\alpha \vee \beta))$
true	true	true	false	true	true
false	true	true	true	true	true
true	false	false	false	false	true
false	false	true	true	true	true

Common rules of substitution

$\alpha \equiv (\alpha \vee \alpha)$	idempotence of $\vee$
$\alpha \equiv (\alpha \wedge \alpha)$	idempotence of $\wedge$
$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$	commutativity of $\vee$
$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$	commutativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$	DeMorgan's Law
$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$	DeMorgan's Law
$(\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$
$(\alpha \supset \beta) \equiv (\neg\beta \supset \neg\alpha)$	contrapositive
$(\alpha \supset \beta) \equiv (\neg\alpha \vee \beta)$	implication
$(\alpha \equiv \beta) \equiv ((\alpha \supset \beta) \wedge (\beta \supset \alpha))$	equivalence
$((\alpha \wedge \beta) \supset \gamma) \equiv (\alpha \supset (\beta \supset \gamma))$	exportation
$((\alpha \supset \beta) \wedge (\alpha \supset \neg\beta)) \equiv \neg\alpha$	absurdity

$(\alpha \vee true) \equiv true$

$(\alpha \vee false) \equiv \alpha$

$(\alpha \wedge true) \equiv \alpha$

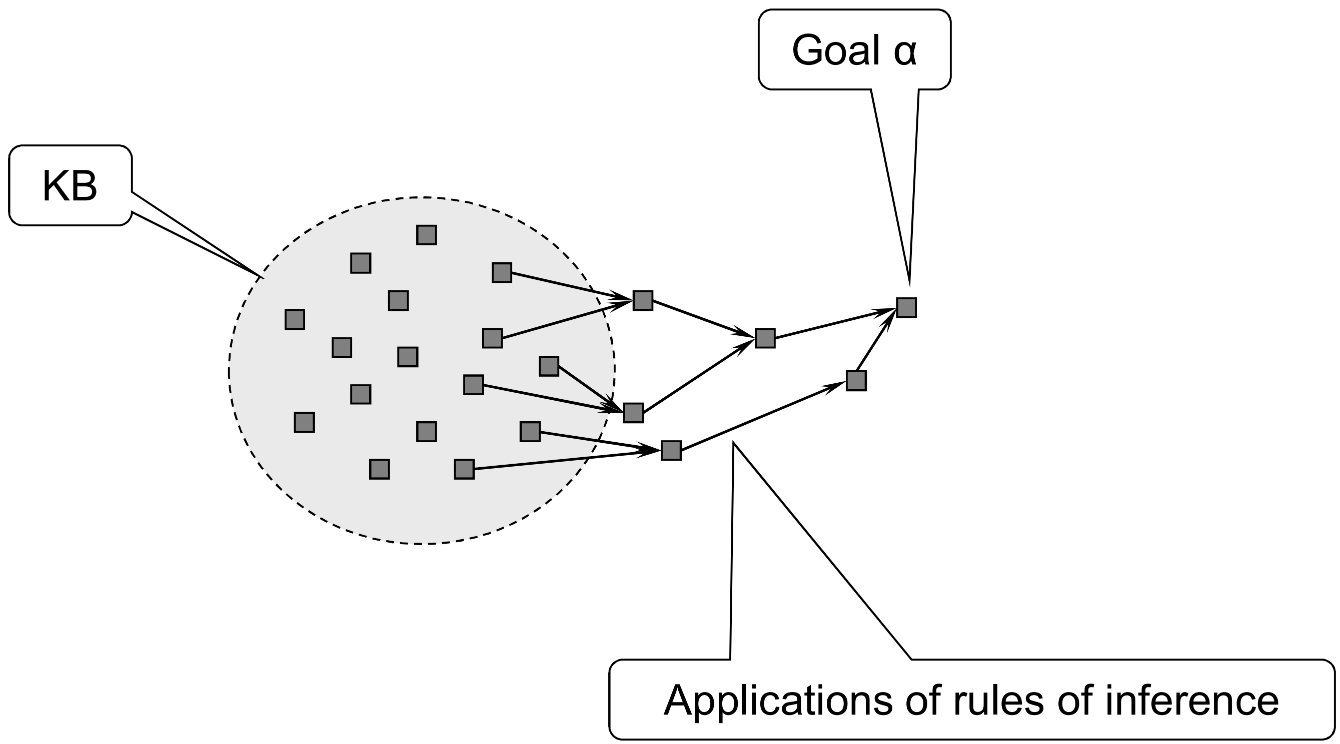
$(\alpha \wedge false) \equiv false$

$(\alpha \wedge \neg\alpha) \equiv false$

$(\alpha \vee \neg\alpha) \equiv true$

$\alpha \equiv \neg(\neg\alpha)$

Proof



Logic: Semantics

Denotation: What does a sentence stand for?

Truth Functional: Is the sentence *true*?

Axioms / Premises (KB): Assumed to be *true*.

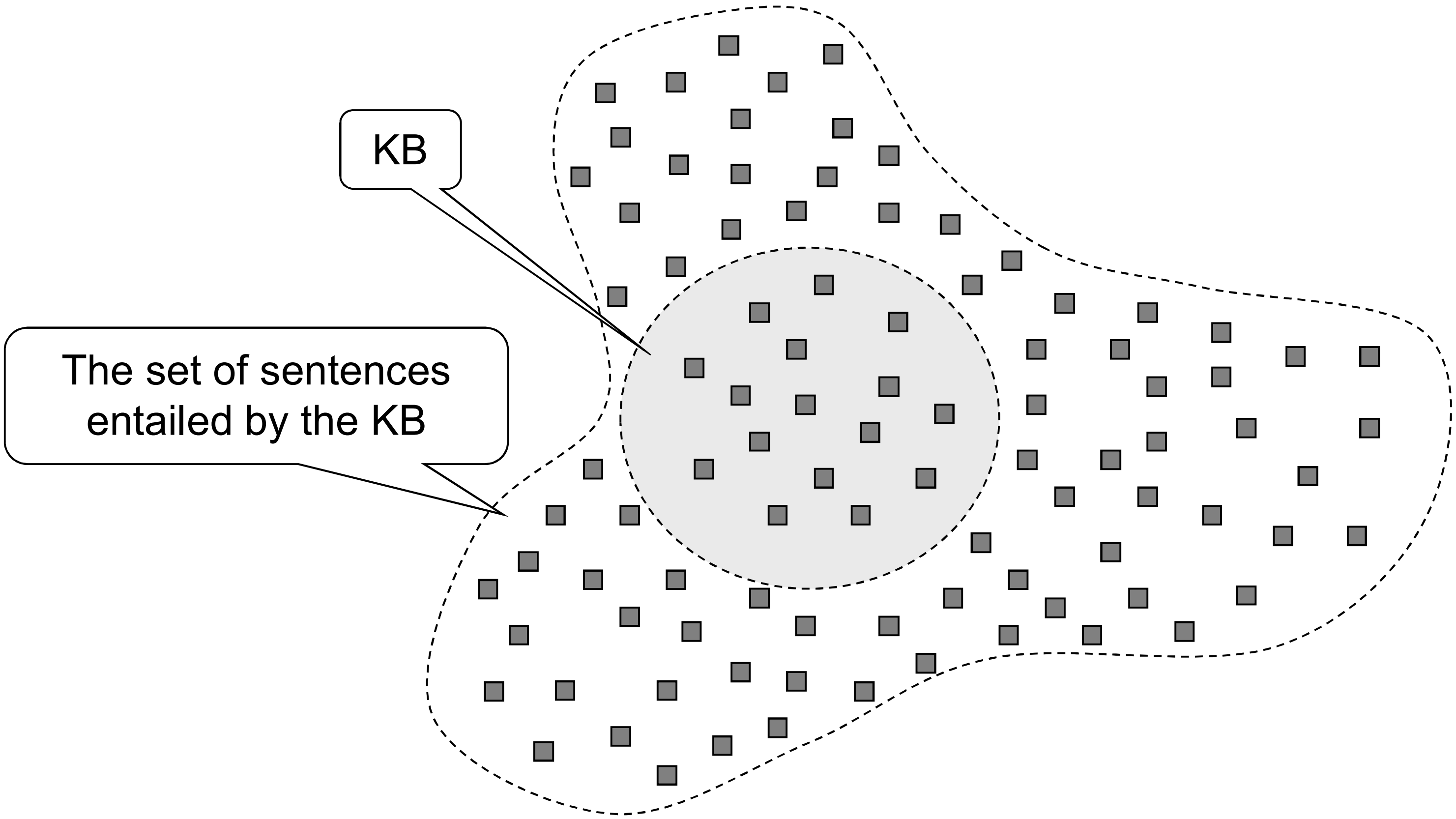
KB is *true* iff every sentence in the KB is *true*.

Entailment: A sentence α is said to be entailed by a set of sentences *S*/KB if the sentence is necessarily *true* whenever S/KB is *true*

$$KB \vDash \alpha$$

We also say that α is *true* (given the KB)

The set of true statements



Soundness and completeness

Given a knowledge base and a reasoning algorithm –

Entailment: which other sentences in the language are necessarily true?

Proof: which other sentences in the language can one produce by the reasoning algorithm?

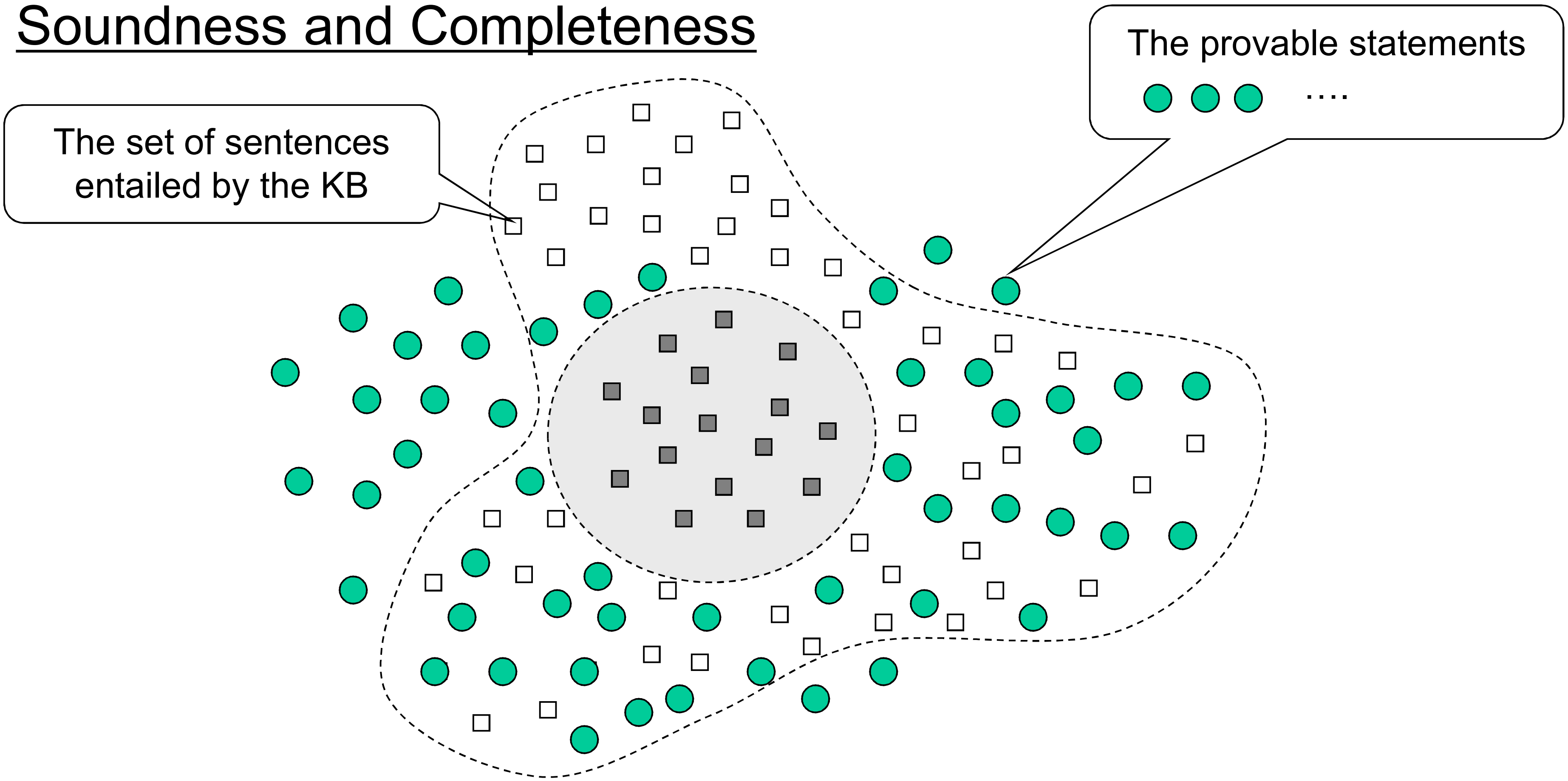
Soundness (of the reasoning algorithm):

A logic is sound if **only** true statements in the language can be proved

Completeness (of the reasoning algorithm):

A logic is complete if **all** true statements in the language can be proved

Soundness and Completeness



First Order Logic (FOL): Syntax

The *logical* part of the vocabulary

- Symbols that stand for connectives or operators
  - "∧", "∨", "~", and "⊃"…
- Brackets "(", ")", "{", "}"…
- The constant symbols "⊥" and "⊤".
- A set of variable symbols $V = \{v_1, v_2, v_3, \ldots\}$
  - commonly used $\{x, y, z, x_1, y_1, z_1, \ldots\}$

•Quantifiers: "∀" read as "for all", and "∃" read as "there exists". The former is the *universal quantifier* and the latter the *existential quantifier*.

- The symbol "=" read as "equals".

FOL Syntax (contd)

The non-logical part of *FOL vocabulary* constitutes of three sets.

- A set of predicate symbols P = {P_1, P_2, P_3, …}. We also use the symbols {*P*, *Q*, R, …}. More commonly we use words like "Man", "Mortal", "GreaterThan". Each symbol has an arity associated with it.
- A set of function symbols F = {f_1, f_2, f_3, …}. We commonly used the symbols {f, g, h…} or words like "Successor" and "Sum". Each function symbol has an arity that denotes the number of argument it takes.
- A set of constant symbols C = {c_1, c_2, c_3, …}. We often used symbols like "0", or "Socrates", or "Darjeeling" that are meaningful to us.

The three sets define a specific language L(*P*,*F,C*).

Terms of L(*P*,*F*,*C*)

The basic constituents of *FOL* expressions are *terms*. The set of terms $\mathfrak{I}$ of L(*P*,*F*,*C*) is defined as follows. The constants and the variables are terms by definition. More terms are defined using the function symbols.

If $t \in V$ then $t \in \mathfrak{I}$

If $t \in C$ then $t \in \mathfrak{I}$

If $t_1, t_2, \ldots, t_n \in \mathfrak{I}$ and $f \in F$ is an n-place function symbol then $f(t_1,t_2, \ldots,t_n) \in \mathfrak{I}$

Atomic Formulas of L(P,F,C)

The set of formulas is defined using terms and predicate symbols. By default the logical symbols "⊥" and "⊤" are also formulas. The set of well formed formulas *F* of L(*P*,*F*,*C*) is defined as follows.

Atomic formulas $\mathcal{A}$

$\bot \in \mathcal{A}$

$\top \in \mathcal{A}$

If $t_1, t_2 \in \Im$ then $(t_1 = t_2) \in \mathcal{A}$

If $t_1, t_2, \ldots, t_n \in \Im$ and $P \in$ P is an n-place predicate symbol then $P(t_1, t_2, \ldots, t_n) \in \mathcal{A}$

Formulas of L(P,F,C)

The set of formulas of L(P,F,C) $\mathcal{F}$ is defined as follows

If $\alpha \in \mathcal{A}$ then $\alpha \in \mathcal{F}$

If $\alpha \in \mathcal{F}$ then $\sim\alpha \in \mathcal{F}$

If $\alpha, \beta \in \mathcal{F}$ then $(\alpha \wedge \beta) \in \mathcal{F}$

If $\alpha, \beta \in \mathcal{F}$ then $(\alpha \vee \beta) \in \mathcal{F}$

If $\alpha, \beta \in \mathcal{F}$ then $(\alpha \supset \beta) \in \mathcal{F}$

Universal and Existential Quantifiers

If $\alpha \in \mathcal{F}$ and $x \in V$ then $\forall x\ (\alpha) \in \mathcal{F}$

$\forall x\ (\alpha)$ is read as "for all x (α)"

If $\alpha \in \mathcal{F}$ and $x \in V$ then $\exists x\ (\alpha) \in \mathcal{F}$

$\exists x\ (\alpha)$ is read as "there exists x (α)"

We will also use the notation (forall (x) (α)) and (exists (x) (α)) as given in the book Artificial Intelligence by Eugene Charniak and Drew McDermott.

Makes representation for use in programs simpler.

List notation

Standard mathematical notation

1. $\forall x\ (Man(x) \supset Human(x))$: all men are human beings
2. $Happy(suresh) \vee Rich(suresh)$: Suresh is rich or happy
3. $\forall x\ (CitrusFruit(x) \supset \neg Human(x))$: all citrus fruits are non-human
4. $\exists x\ (Man(x) \wedge Bright(x))$: some men are bright

List notation (a la Charniak & McDermott, "Artificial Intelligence")

1.(forall (x) (if (man x) (human x)))

2.(or (happy suresh) (rich suresh))

3.(forall (x) (if (citrusFruit x) (not (human x))))

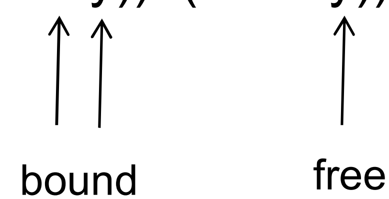
4.(exists (x) (and (man x) (bright x)))

Sentences of L(P,F,C)

A variable within the scope of a a quantified is said to be *bound*.

If a variable is not bound then it is *free*.

Example: (forall (x) (and (exists (y) (loves-to-read x y)) (book y))



A formula of L(P,F,C) without free variables
is a sentence of L(P,F,C).

FOL: Rules of Inference

The propositional logic rules we saw earlier are valid in *FOL* as well. In addition we need new rules to handle quantified statements. The two commonly used rules of inference are,

$$\frac{\forall x\ P(x)}{P(a)} \quad \text{where } a \in C \qquad \text{Universal Instantiation (UI)}$$

$$\frac{P(a)}{\exists x\ P(x)} \quad \text{where } a \in C \qquad \text{Generalization}$$

Examples:

$$\frac{\forall x\ (Man(x) \supset Mortal(x))}{(Man(Socrates) \supset Mortal(Socrates))}$$

$$\frac{(Man(Socrates) \supset Mortal(Socrates))}{\exists x\ (Man(x) \supset Mortal(x))}$$

FOL: Rules of Substitution

The following rules of substitution are also useful,

$\neg\forall x\ \alpha$	$\equiv$	$\exists x\ \neg\alpha$	DeMorgan's law
$\neg\exists x\ \alpha$	$\equiv$	$\forall x\ \neg\alpha$	DeMorgan's law
$\forall x\ \forall y\ \alpha$	$\equiv$	$\forall y\ \forall x\ \alpha$	
$\exists x\ \exists y\ \alpha$	$\equiv$	$\exists y\ \exists x\ \alpha$	

Semantics (Propositional Logic)

Atomic sentences in Propositional Logic can stand for anything. Consider,

Alice likes mathematics and she likes stories. If she likes mathematics she likes algebra. If she likes algebra and likes physics she will go to college. She does not like stories or she likes physics. She does not like chemistry and history.

Encoding: P = Alice likes mathematics. Q = Alice likes stories. R = Alice likes algebra. S = Alice likes physics. T = Alice will go to college. U = Alice likes chemistry. V = Alice likes history.

Then the given facts are,

$(P \wedge Q)$

$(P \supset R)$

$((R \wedge S) \supset T)$

$(\sim Q \vee S)$

$(\sim U \wedge \sim V)$

Semantics (First Order Logic)

Difficult to express universal statements meaningfully in Propositional Logic. Consider,

Alice likes mathematics and she likes stories. If she likes mathematics she likes algebra. If she likes algebra and likes physics she will go to college. …

The statements in red colour are specific to Alice. We often want to make these as general statements - *If SOMEONE likes mathematics she likes algebra. If SOMEONE likes algebra and likes physics she will go to college.*

To make such general statements and reason with them we need the notion of variables that FOL gives us, and the universal and existential quantifiers. The variables take values from a domain, and thus we have the notion of Interpretations in FOL where we choose a domain and interpret the language L(P,F,C) over the domain.

Semantics: Interpretations for L(P,F,C)

An Interpretation ϑ = <D, I> of a FOL language L(P,F,C) constitutes of a domain (or Universe of Discourse) *D* and a mapping *I* from the language *L* to the domain *D*.

Each of the elements of the sets *P, F* and *C* are interpreted over *D*. Each of them is understood or gets meaning from the domain *D*.

Predicate symbols mapped to relations on *D*

Function symbols mapped to functions on *D*

Constant symbols mapped to individuals in *D*

Interpretation ϑ = <D,I> of L(P,F,C)

For every predicate symbol $Q \in P$ of arity N,

$I(Q) = Q^I$ where Q^I is the image of Q and $Q^I \subseteq DxDx\ldots xD$

For every function symbol $f \in F$ of arity N,

$I(f) = f^I$ where f^I is the image of f and f^I: $DxDx\ldots xD \rightarrow D$

For every constant symbol $c \in C$

$I(c) = c^I$ where c^I is the image of c and $c^I \in D$

In addition we have an assignment A: $V \rightarrow D$ from the set of variables of L(P,F,C) to the domain.

$A(v) = v^A$ where $v^A \in D$

Interpretation of Terms of L(P,F,C)

Terms in FOL *denote* elements in the domain.

A term $t \in \mathfrak{I}$ mapped to the element of the domain *D* as follows.

If $t \in V$ then $t^{IA} = t^{A}$

If $t \in C$ then $t^{IA} = t^{I}$

If $t = f(t_1, t_2, \ldots, t_n)$ and $f \in F$ then $t^{IA} = f^{I}(t_1^{IA}, t_2^{IA}, \ldots, t_n^{IA})$

Variables are mapped by the assignment A. For example, x → 12

Constants are interpreted by the mapping I. For example, sifar → 0

Functions denote elements too. For example, plus(3,8) → 11

Truth Assignment to Atomic Formulas of FOL

A valuation function Val: $\mathcal{F} \rightarrow$ {*true, false*}

Val(⊤) = *true*

Val(⊥) = *false*

$\text{Val}(t_1=t_2)^{IA} = \textit{true} \quad \text{iff} \quad t_1^{IA} = t_2^{IA}$

$\text{Val}(Q(t_1,t_2,\ldots,t_n))^{IA} = \text{true iff} < t_1^{IA}, t_2^{IA}, \ldots , t_n^{IA} > \in Q^{I}$

For example,

- colour(lily) = white is *true* iff both refer to the same colour
- president(usa) = commander(us_army) is *true* iff both refer to the same person
- LessThan(5,17) is *true* iff <5,17> ∈ < relation on Natural Numbers
- Brother(suresh, ramesh) is *true* iff <suresh,rames> ∈ Brother relation on the set of people

Truth Assignment to Formulas of FOL

Logical connectives are interpreted in the standard way

If Val(α) = *true* then Val(¬α) = *false*

If Val(α) = *false* then Val(¬α) = *true*

If Val(α) = *false* and Val(β) = *false* then Val(α∨β) = *false*
else Val(α∨β) = *true*

If Val(α) = *true* and Val(β) = *false* then Val(α⊃β) = *false*
else Val(α⊃β) = *true*

If Val(α) = *true* and Val(β) = *true* then Val(α∧β) = *true*
else Val(α∧β) = *false*

Truth Assignment to Quantified Formulas of FOL

A formula of the form $\exists x(\alpha)$ is true if there is some value of *x* for which the formula is true. A universally quantified formula $\forall x\ (\alpha)$ is true for all possible values of *x*. Formally,

$(\exists x\ (\alpha))^{IA}$ = true iff α^{IB} is true for *some* assignment B that is an x-variant of A.
In other words the formula α is true for some value of *x*.

$(\forall x\ (\alpha))^{IA}$ = true iff α^{IB} is true for *all* assignments B that are x-variants of A.
In other words the formula α is true for all values of *x*.

An assignment *B* is said to be an x-variant of an assignment *A* if they agree on the value of all variables except *x*.

Truth Assignment to Sentences of FOL

- A sentence in FOL is a formula without any free variables.
- This means that all variables in the formula are quantified.
- As a consequence the sentences are *true* or *false* independent of the assignment mapping.

The meaning of the terms and sentences of a set of *FOL* sentences is given by an *interpretation* ϑ = <D, I>, where *D* is a domain and *I* is an interpretation mapping.

An interpretation M = <D, I> of set of sentences or a theory in a language L(*P*,*F*,*C*) is a *model* if all the sentences in the set are *true* in the interpretation.

Tautologies, Satisfiable and Unsatisfiable formulas

- A **tautology** is a formula of L(P,F,C) that is *true* in all interpretations.
  - For example Happy(suresh) $\vee \neg$ Happy(suresh)
- A formula of L(P,F,C) is **satisfiable** iff it is *true* in at least one interpretation.
  - For example $\forall$x (Man(x) $\supset$ Human(x))
    (depends on the meaning of Man and Human)
- A formula of L(P,F,C) is **unsatisfiable** iff it is *true* in no interpretation.
  - For example Happy(suresh) $\wedge \neg$ Happy(suresh)

End of Module 1